PUTNAM PRACTICE SET 4

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Problem 1. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be defined as follows: for each positive integer n, we let $f(n)$ be the sum of the digits of n. Find

$$f\left(f\left(f\left(2018^{2018}\right)\right)\right).$$

Problem 2. Let $x, y \in \mathbb{R}$ such that $x + y = 1$. Prove that

$$x^{m+1} \cdot \left(\sum_{j=0}^{n} \binom{m+j}{j} y^j\right) + y^{n+1} \cdot \left(\sum_{i=0}^{m} \binom{n+i}{i} x^i\right) = 1,$$

for each $m, n \in \mathbb{N}$.

Problem 3. For any real numbers $a < b$ and for any continuous function $g : [a, b] \longrightarrow \mathbb{R}$, we denote by $G(g)$ the graph of $g(x)$, i.e., the set

$$G(g) := \{(x, y) \colon a \le x \le b \text{ and } y = g(x)\}.$$

Also, for any function $f : [a, b] \longrightarrow \mathbb{R}$ and for any $c \in \mathbb{R}$, we denote by f_c the function $[a + c, b + c] \longrightarrow \mathbb{R}$ given by $f_c(x) := f(x - c)$. Find with proof all the real numbers $c \in (0, 1)$ with the property that there exists some continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ (depending on c) such that:

- $f(0) = f(1) = 0$; and
- $G(f)$ and $G(f_c)$ are disjoint.

Problem 4. Find all polynomials $P \in \mathbb{R}[x, y]$ satisfying the following properties:

(1) there exists some $n \in \mathbb{N}$ with the property that $P(tx, ty) = t^n P(x, y)$ for all $t, x, y \in \mathbb{R}$;
(2) $P(a + b, c) + P(b + c, a) + P(c + a, b) = 0$ for all $a, b, c \in \mathbb{R}$; and
(3) $P(1, 0) = 1$.

Problem 5. Let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. Prove that there exist infinitely many $n \in \mathbb{N}$ for which there exist $(k, m, x, y) \in \mathbb{N}^4$ such that $a_n = xa_k + ya_m$.

Problem 6. Let $0 < x_1 < \cdots < x_n < \frac{\pi}{2}$ be real numbers. Prove that:

$$\sum_{i=1}^{n-1} \sin(2x_i) - \sum_{i=1}^{n-1} \sin(x_i - x_{i+1}) < \frac{\pi}{2} + \sum_{i=1}^{n-1} \sin(x_i + x_{i+1}).$$